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On the numerical index of vector-valued function spaces

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Let X be a Banach space and μ a positive measure. In this article, we show that $n(L_p(\mu, X)) = \lim_m n(l_p^m(X))$, $1 \le p < \infty$. Also, we investigate the positivity of the numerical index of l_p-spaces.

Keywords: Numerical index; Numerical radius

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1. Introduction

Let X be a Banach space over $\mathbb{R}$ or $\mathbb{C}$, we write B_X for the closed unit ball and S_X for the unit sphere of X. The dual space is denoted by X^* and the Banach algebra of all continuous linear operators on X is denoted by $B(X)$. The *numerical range* of $T \in B(X)$ is defined by

$$V(T) = \left\{x^*(Tx)\colon x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\right\}\cdot$$

The *numerical radius* of T is then given by

$$v(T) = \sup\left\{|\lambda|\colon \lambda \in V(T)\right\}\cdot$$

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Clearly, v is a semi norm on $B(X)$ and $v(T) \leq \|T\|$ for all $T \in B(X)$. The *numerical index* of X is defined by

$$n(X) = \inf\{v(T)\colon T \in S_{B(X)}\}\cdot$$

The concept of numerical index was first suggested by Lumer [7] in 1968. Since then a lot of attention has been paid to this constant of equivalence between the numerical radius and the usual norm in the Banach algebra of all bounded linear operators of a Banach space. Classical references here are [1,2]. For recent results, we refer the reader to [3,5,6,8,10].

In this article, we show that for any positive measure μ and Banach space X, the numerical index of $L_p(\mu, X)$, $1 \leq p < \infty$ is the limit of the sequence of numerical index of $l_p^m(X)$. This gives a partial answer to Martín's question [9] and generalizes the result obtained for the scalar case [5]. We also study the positivity of the numerical index of l_p-space.

Here, $L_p(\mu, X)$ is the classical Banach space of p-integrable functions f from Ω into X where (Ω, Σ, μ) is a given measure space. And $l_p(X)$ is the Banach space of sequences $x = (x_n)_{n\geq 1}$, $x_n \in X$, such that $\sum_{n=1}^{\infty} \|x_n\|^p < \infty$. Finally $l_p^m(X)$ is the Banach space of finite sequences $x = (x_n)_{1\leq n\leq m}$, $x_n \in X$, equipped with the norm $\|x\|_p = (\sum_{n=1}^{m} \|x_n\|^p)^{1/p}$.

2. Main results

Theorem 2.1 *Let X be a Banach space. Then, for every real number $p, 1 \leq p < \infty$, the numerical index of the Banach space $l_p(X)$ is given by*

$$n(l_p(X)) = \lim_m n\big(l_p^m(X)\big).$$

Proof Let $m \geq 1$ and $T : l_p^m(X) \to l_p^m(X)\ x \mapsto (T_1(x), \ldots, T_m(x))$. Define the linear operator $\tilde{T} : l_p(X) \to l_p(X)$ as follows for $x = (x_1, \ldots, x_m, x_{m+1}, \ldots) \in l_p(X)$, $\tilde{T}(x) = (T_1(x_1, \ldots, x_m), \ldots, T_m(x_1, \ldots, x_m), 0, \ldots)$. Clearly, $\tilde{T}$ is bounded and $\|T\| = \|\tilde{T}\|$. We have also $v(T) = v(\tilde{T})$. To prove this, let us first note that if $x = (x_1, \ldots, x_m, \ldots) \in S_{l_p(X)}$, then there exists an element, namely x_x^*, in $S_{l_q(X^*)}$, where q is the conjugate exponent to p, such that $x_x^*(x) = 1$. Explicitly $x_x^* = (\|x_1\|^{p-1} x_1^*, \ldots, \|x_m\|^{p-1} x_m^*, \ldots)$ where the x_k^*'s are taken in S_{X^*} such that $x_k^*(x_k) = \|x_k\|$. Now, let $\varepsilon > 0$. Following the expression $v(\tilde{T}) = \sup\{|x_x^*(\tilde{T}x)| : x \in S_{l_p(X)}\}$ ([4], Lemma 3.2 and Proposition 1.1) there exists $x = (x_1, \ldots, x_m, x_{m+1}, \ldots) \in S_{l_p(X)}$ such that

$$\begin{aligned} v(\tilde{T}) - \varepsilon &< |x_x^*(\tilde{T}x)| \\ &= \big|\big(\|x_1\|^{p-1} x_1^*, \ldots, \|x_m\|^{p-1} x_m^*\big)\big(T(x_1, \ldots, x_m)\big)\big|. \end{aligned}$$

Put $r := (\sum_{k=1}^{m} \|x_k\|^p)^{1/p} \leq 1$. Then we obtain $v(\tilde{T}) - \varepsilon < r^p v(T)$ which yields $v(\tilde{T}) \leq v(T)$. The reverse inequality is easy. Therefore

$$\Big\{v(T)\colon T \in l_p^m(X), \|T\| = 1\Big\} \subset \{v(U) : U \in l_p(X), \|U\| = 1\}$$

which yields $n(l_p(X)) \le n(l_p^m(X))$. Consequently $n(l_p(X)) \le \liminf_m n(l_p^m(X))$. Now we shall prove that $\limsup_m n(l_p^m(X)) \le n(l_p(X))$. Let $T \in B(l_p(X))$. Define the sequence of operators $\{S_m\}_m$ as follows; for each $m \ge 1$, S_m is defined on $l_p^m(X)$ by

$$S_m(x) = (T_1(x_1, \dots, x_m, 0, 0, \dots), \dots, T_m(x_1, \dots, x_m, 0, 0, \dots)) \quad \left(x \in l_p^m(X)\right)\cdot$$

Clearly, the S_m's are bounded and $\|S_m\| \le \|T\|$ for all m. We claim that

(i) $\|S_m\| \to \|T\|$
(ii) $v(S_m) \to v(T)$.

Indeed, we consider the sequence of operators $\{\tilde{S}_m\}_m$ defined on $l_p(X)$ by

$$\tilde{S}_m(x) = (T_1(x_1, \dots, x_m, 0, 0, \dots), \dots, T_m(x_1, \dots, x_m, 0, 0, \dots), 0, 0, \dots)$$

for all $x = (x_1, \dots, x_m, x_{m+1}, \dots) \in l_p(X)$. It is easy to see that $\|S_m\| = \|\tilde{S}_m\|$, and $\tilde{S}_m$ converges strongly to T. This implies that $\|T\| \le \liminf_m \|\tilde{S}_m\|$, and it follows that $\|S_m\| \to \|T\|$. As in (i) we have also $v(S_m) = v(\tilde{S}_m)$, so it is enough to prove that $v(\tilde{S}_m) \to v(T)$. Let $\varepsilon > 0$ and fix $u \in S_X$, $u^* \in S_{X^*}$ such that $u^*(u) = 1$. There exists $x \in S_{l_p(X)}$ such that

$$\left|x_x^*(Tx)\right| > v(T) - \varepsilon\cdot \tag{1}$$

For each $n \ge 1$, consider

$$x^n = \left(x_1, \dots, x_{n-1}, \lambda_n u, 0, 0, \dots\right); \quad x_{x^n}^* = \left(\|x_1\|^{p-1} x_{x_1}^*, \dots, \|x_{n-1}\|^{p-1} x_{x_{n-1}}^*, \lambda_n^{p-1} u^*, 0, 0, \dots\right)$$

where $\lambda_n = \left(\sum_{k=n}^{\infty} \|x_k\|^p\right)^{1/p}$. Then

$$x_{x^n}^*(x^n) = 1 = \|x_{x^n}^*\| = \|x^n\|\cdot$$

Moreover, $\|x - x^n\| \to 0$ and $\|x_x^* - x_{x^n}^*\| \to 0$ where $x_x^* = \left(\|x_1\|^{p-1} x_{x_1}^*, \dots, \|x_n\|^{p-1} x_{x_n}^*, \dots\right)$. It follows that $x_{x^n}^*(Tx^n) \to x_x^*(Tx)$ as n tends to infinity. Let $n_0 \ge 1$ be such that

$$|x_{x^n}^*(Tx^n)| > v(T) - \varepsilon \quad (n \ge n_0)\cdot \tag{2}$$

Since $\tilde{S}_m$ converges strongly to T, thus for fixed $n \ge n_0$, $x_{x^n}^*(\tilde{S}_m x^n)$ converges to $x_{x^n}^*(Tx^n)$ as m tends to infinity. So there is $m_0 \ge n$ such that

$$|x_{x^n}^*(\tilde{S}_m x^n)| > v(T) - \varepsilon \quad (m \ge m_0)\cdot \tag{3}$$

This yields $v(\tilde{S}_m) > v(T) - \varepsilon$ for all $m \ge m_0$ and therefore $v(\tilde{S}_m)$ converges to $v(T)$ as m tends to infinity. Now, following (i) and (ii) we have $n(l_p(X)) \ge \limsup_m n(l_p^m(X))$. Indeed, for a given $\varepsilon > 0$, we find $T \in S_{B(l_p(X))}$ such that

$$n(l_p(X)) + \varepsilon > v(T)\cdot$$

Since $v(T) = \lim_m v(\tilde{S}_m)$, there exists m_0 such that

$$n(l_p(X)) + \varepsilon > v(\tilde{S}_m) \quad (m \geq m_0)\cdot$$

But $v(\tilde{S}_m) = v(S_m) \geq n(l_p^m(X))\|S_m\|$, and $\|S_m\| \to \|T\| = 1$, so there exists $k_0 \geq m_0$ such that

$$n(l_p(X)) + \varepsilon > n(l_p^m(X))(1 - \varepsilon) \quad (m \geq k_0)\cdot$$

This implies $n(l_p(X)) \geq \limsup_m n(l_p^m(X))$ and completes the proof of Theorem 2.1. ■

It is well known that $n(\oplus_\lambda X_\lambda)_{l_\infty} = \inf_{\lambda \in \Lambda} n(X_\lambda)$ [9]. This shows that, in particular, $n(l_\infty(X)) = n(X)$ $(= \lim_m n(l_\infty^m(X)))$. So, Theorem 2.1 is also valid for $p = \infty$.

THEOREM 2.2 *Let* (Ω, Σ, μ) *be a* σ*-finite measure space. Then, for every Banach space* X *and every real number* $p, 1 \leq p < \infty$,

$$n(L_p(\mu, X)) = n(l_p(X))\cdot$$

Proof Let us first prove that $n(L_p(\mu, X)) \leq n(l_p(X))$. For this we adapt the proof due to Javier and Martin for the scalar case (unpublished result). Indeed, if μ is not atomic, $L_p(\mu, X)$ is isometric to $L_p(\mu, X) \oplus_p L_p(\mu, X)$, so they have the same numerical index. Let $T = (T_1, T_2) \in B(l_p^2(X))$ and define the operator S on $L_p(\mu, X) \oplus_p L_p(\mu, X)$ by $S(f_1, f_2)(\omega) = T(f_1(\omega), f_2(\omega))$. One can check easily that $\|T\| = \|S\|$. Moreover, $v(T) = v(S)$. Indeed, let $f_1 = \sum_{i=1}^m x_i(1_{A_i}/\mu(A_i)^{1/p})$, $f_2 = \sum_{i=1}^n y_i(1_{B_i}/\mu(B_i)^{1/p})$ be simple functions in $L_p(\mu, X)$ with $\|(f_1, f_2)\|^p = \sum_{i=1}^m \|x_i\|^p + \sum_{i=1}^n \|y_i\|^p = 1$. For each i we can find x_i^* and y_i^* in S_{X^*} such that $x_i^*(x_i) = \|x_i\|$ and $y_i^*(y_i) = \|y_i\|$. If we set $g_1 = \sum_{i=1}^m \|x_i\|^{p-1} x_i^*(1_{A_i}/\mu(A_i)^{1/q})$ and $g_2 = \sum_{i=1}^n \|y_i\|^{p-1} y_i^*(1_{B_i}/\mu(B_i)^{1/q})$, we have clearly $(g_1, g_2) \in S_{L_q(\mu, X^*) \oplus_q L_q(\mu, X^*)}$ and $< (g_1, g_2), (f_1, f_2) >= 1$. Moreover,

$$\begin{aligned} |(g_1, g_2)(S(f_1, f_2))| &\leq \int_\Omega |(g_1(\omega), g_2(\omega))(T(f_1(\omega), f_2(\omega)))| \mathrm{d}\mu(\omega) \\ &\leq v(T) \int_\Omega \|f_1(\omega)\|^p + \|f_2(\omega)\|^p \mathrm{d}\mu(\omega) = v(T). \end{aligned}$$

Following [4], we have $v(S) \leq v(T)$. For the reverse inequality, let $(x_1, x_2) \in S_{l_p^2(X)}$. Take $A \in \Sigma$ with $\mu(A) > 0$ and consider $(f_1, f_2) = (x_1(1_A/\mu(A)^{1/p}), x_2(1_A/\mu(A)^{1/p}))$. From what we have just seen $(g_1, g_2) = (\|x_1\|^{p-1} x_1^*(1_A/\mu(A)^{1/q}), \ \|x_2\|^{p-1} x_2^*(1_A/\mu(A)^{1/q})) \in S_{L_q(\mu, X^*) \oplus_q L_q(\mu, X^*)}$ and $< (g_1, g_2), (f_1, f_2) >= 1$. Moreover,

$$\left|(\|x_1\|^{p-1} x_1^*, \|x_2\|^{p-1} x_2^*)(T(x_1, x_2))\right| = \left| \int_\Omega (g_1(\omega), g_2(\omega)) S(f_1, f_2)(\omega) \mathrm{d}\mu(\omega) \right| \leq v(S)\cdot$$

This yields $v(T) \leq v(S)$. Consequently $\{v(T)\colon T \in S_{l_p^2(X)}\} \subset \{v(S)\colon S \in S_{L_p(\mu, X) \oplus_p L_p(\mu, X)}\}$ which yields $n(L_p(\mu, X) \oplus_p L_p(\mu, X)) \leq n(l_p^2(X))$. So

$$n(L_p(\mu, X)) \leq n(l_p^2(X))\cdot$$

Now, for any integer $m \geq 1$, with the same work as above, we obtain

$$n(L_p(\mu, X)) \leq n(l_p^m(X))\cdot$$

It follows from Theorem 2.1 that

$$n(L_p(\mu, X)) \leq n(l_p(X))\cdot$$

If μ is atomic then $L_p(\mu, X)$ is isometric to $L_p(\nu, X) \oplus_p \left[\oplus_{i\in I} X\right]_{l_p}$ for a suitable set I and an atomless measure ν. With the help of Remark 2 [9], we also have $n(L_p(\mu, X)) \leq n(l_p(X))$. The reverse inequality $n(L_p(\mu, X)) \geq n(l_p(X))$ follows with the same technique used in [5] for the scalar case. ■

Corollary 2.3 *Let (Ω, Σ, μ) be a σ-finite measure space. Then, for every Banach space X and every real number $p, 1 \leq p < \infty$*

$$n(L_p(\mu, X)) = \lim_m n\big(l_p^m(X)\big)\cdot$$

3. On the positivity of the numerical index of l_p-space

It was proved that the numerical index of l_p^m, $p \neq 2$, $m = 1, 2, \ldots$ cannot be equal to 0 this is equivalent to that the numerical radius and the operator norm are equivalent on $B(l_p^m)$, $p \neq 2$ (see Theorem 2.3 [6]). In this section we shall also prove that both norms are equivalent on $B(l_p, l_p^m)$.

Theorem 3.1 *For every real number $p \geq 1, p \neq 2$ and every integer m, the numerical radius is equivalent to the operator norm on $B(l_p, l_p^m)$.*

Here l_p is real and l_p^m is identified with its natural embedding in l_p.

Proof Let $T = (t_{ik}) \in B(l_p, l_p^m)$. We first have

$$\|T\| \leq \left\| \left(\sum_{k=1}^{\infty} |t_{1k}|^q\right)^{1/q}, \ldots, \left(\sum_{k=1}^{\infty} |t_{mk}|^q\right)^{1/q} \right\|_p$$

$$\leq \left(\sum_{k=1}^{\infty} |t_{1k}|^q\right)^{1/q} + \cdots + \left(\sum_{k=1}^{\infty} |t_{mk}|^q\right)^{1/q}.$$

Consider $\{T^j\} \in B(l_p, l_p^m)$ defined by $T^j e_k = Te_k$ for $k \neq j$ and $T^j(e_j) = 0$. Then for $x = \sum_{k=1}^{\infty} x_k e_k \in S_{l_p}$ we have

$$x_x^*(T^1 x) = \varepsilon_1 |x_1|^{p-1} \sum_{k=2}^{\infty} t_{2k} x_k + \cdots + \varepsilon_m |x_m|^{p-1} \sum_{k=2}^{\infty} t_{mk} x_k \quad (\varepsilon_j \in \{-1, 1\})\cdot$$

Take $x_1 = \varepsilon_1 2^{-1/p}$ with $\varepsilon_1 \in \{-1, 1\}$ we obtain

$$\left|x_x^*(T^1 x)\right| = \left|2^{-1/q}\left(\sum_{k=2}^{\infty} t_{1k}x_k\right) + \varepsilon_1\left\{\varepsilon_2|x_2|^{p-1}\sum_{k=2}^{\infty} t_{2k}x_k + \cdots + \varepsilon_m|x_m|^{p-1}\sum_{k=2}^{\infty} t_{mk}x_k\right\}\right| \leq v(T^1)$$

Since ε_1 is arbitrary in $\{-1, 1\}$ then

$$2^{-1/q}\left|\sum_{k=2}^{\infty} t_{1k}x_k\right| + \left|\varepsilon_2|x_2|^{p-1}\sum_{k=2}^{\infty} t_{2k}x_k + \cdots + \varepsilon_m|x_m|^{p-1}\sum_{k=2}^{\infty} t_{mk}x_k\right| \leq v(T^1).$$

And in particular

$$2^{-1/q}\left|\sum_{k=2}^{\infty} t_{1k}x_k\right| \leq v(T^1)$$

for all $(x_2, \ldots, x_m, \ldots) \in l_p$ such that

$$\sum_{k=2}^{\infty} |x_k|^p = \frac{1}{2}.$$

That is

$$\frac{1}{2}\left|\sum_{k=2}^{\infty} t_{1k}y_k\right| \leq v(T^1) \quad \forall (y_2, \ldots, y_m, \ldots) \in S_{l_p}$$

which yields

$$\frac{1}{2}\Big(\sum_{k\neq 1} |t_{1k}|^q\Big)^{1/q} \leq v(T^1).$$

The same work as above shows that

$$\frac{1}{2}\left(\sum_{k\neq j} |t_{jk}|^q\right)^{1/q} \leq v(T^j) \tag{$*$}$$

for $j = 1, 2, \ldots, m$. Now let $R^j = T - T^j$ then we have

$$v(T^j) \leq v(T) + \|R^j\|.$$

And following (*) we obtain

$$\left(\sum_{k=1}^{\infty} |t_{jk}|^q\right)^{1/q} \leq 2\big(v(T) + \|R^j\|\big) + |t_{jj}|$$

which yields

$$\|T\| \leq 2mv(T) + 2\sum_{j=1}^{m} \|R^j\| + \sum_{j=1}^{m} |t_{jj}|.$$

Now let $\{T_n\}$ be a v-Cauchy sequence in $B(l_p, l_p^m)$. Since $v(T_nP_m) = v(P_mT_nP_m) \leq v(T_n)$ where P_m is the operator projection on l_p^m (see [5] p. 4), and using the fact that in finite dimensional space l_p^m both norms are equivalent, then each $R_n^j = T_n - T_n^j$ converges in operator norm to some R^j. Therefore $\{T_n\}$ is $\|\|$-Cauchy. This completes the proof of the Theorem 3.1. ■

It is still unknown if the numerical radius and the operator norm are equivalent on the Banach space $B(l_p)$, $p \neq 2$ which gives a complete answer to the question of C. Finet and D. Li.

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